Supplementary material for "Strong dispersive coupling of a high finesse cavity to a micromechanical membrane"

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This supplemental material provides a detailed calculation of the quantum nondemolition measurement of a membrane's phonon number.

The purpose of this section is to estimate the feasibility of observing the quantum jumps of a micromechanical device using the setup described in our paper. To do this, we need to estimate three quantities: the cavity frequency shift per membrane phonon (i.e., the signal); the sensitivity with which the cavity frequency can be measured (i.e., the noise spectral density); and the lifetime of a phonon-number state (i.e., the allowable averaging time). Together these three quantities give the signal-to-noise ratio for observing a quantum jump. We assume throughout that the membrane has been cooled to its ground state and the cooling laser switched off.

The shift in the cavity frequency per phonon in the membrane. The Hamiltonian for the optomechanical device is (excluding damping and driving terms):

$$\begin{aligned}\hat{H} &= \hat{N}\hbar\omega_{\text{cav}}(x) + \hat{n}\hbar\omega_m \\ \omega_{\text{cav}}(x) &= \frac{c}{L}\cos^{-1}(r_c\cos(4\pi x/\lambda))\end{aligned} \tag{1}$$

Here $\hat{N}$ is the photon number operator, $\omega_{\text{cav}}(x)$ is the cavity frequency as a function of membrane displacement, $\hat{x}$ is the membrane displacement, $\hat{n}$ is the phonon number operator, ω_m is the membrane's natural frequency and r_c is its field reflectivity.

For QND measurements we want to operate near an extremum in $\omega_{\text{cav}}(x)$. Expanding about some equilibrium membrane position $x_0 \approx 0$ (x_0 is a constant) we have:

$$\omega_{\text{cav}}(x) \approx \omega_{\text{cav},0} + \omega'_{\text{cav},0}(x - x_0) + \omega''_{\text{cav},0}(x - x_0)^2/2\,, \tag{2}$$

where to lowest nonvanishing order in x_0

$$\omega_{\gamma,0} = c\cos^{-1}(r_c)/L \tag{3}$$

$$\omega'_{\gamma,0} = \frac{16\pi^2 c r_c}{L\lambda^2\sqrt{1-r_c^2}}x_0 \tag{4}$$

$$\omega''_{\gamma,0} = \frac{16\pi^2 c r_c}{L\lambda^2\sqrt{1-r_c^2}} \tag{5}$$

Note that if the membrane is positioned precisely at an extremum in $\omega_{\text{cav}}(x)$ (i.e., at x_0 = 0) then $\omega'_{\gamma,0} = 0$ and we have just quadratic detuning.

Now we identify $(x - x_0)$ as the dynamical variable describing the membrane displacement and quantize it to become $\hat{x}$. For the present we assume $x_0 = 0$ exactly, so we can ignore the linear detuning.

Then substituting

$$\hat{x}^2 = x_\mathrm{m}^2(\hat{b}^\dagger + \hat{b})^2 \ , \tag{6}$$

where $x_m = \sqrt{\hbar / 2m\omega_m}$ is the zero-point amplitude of the membrane, gives

$$\hat{H} = \hat{N}\hbar\left(\omega_{\gamma,0} + \tfrac{1}{2}\omega''_{\gamma,0}x_\mathrm{m}^2(\hat{b}^\dagger + \hat{b})^2\right) + \hat{n}\hbar\omega_m \tag{7}$$

If the cavity line is narrow enough to make the $\hat{b}^{\dagger 2}$ & $\hat{b}^2$ terms in (7) irrelevant (i.e., in the rotating wave approximation (RWA)), this becomes

$$\hat{H} = \hat{N}\hbar\left(\omega_{\gamma,0} + \Delta\omega_\gamma(\hat{b}^\dagger\hat{b} + \tfrac{1}{2})\right) + \hat{n}\hbar\omega_m \tag{8}$$

where $\Delta\omega_\gamma$ is the cavity shift per phonon. For $r_\mathrm{c} \sim 1$ this is given by:

- $$\Delta\omega_\gamma = \omega''_{\gamma,0}x_m^2 = \frac{8\pi^2 c}{L\lambda^2\sqrt{2(1-r_c)}}\frac{\hbar}{m\omega_m} . \tag{9}$$

The shot-noise limited frequency resolution of the Pound-Drever-Hall scheme leads to an angular frequency noise power spectral density[1] (i.e., in units of $s^{-2}Hz^{-1}$):

- $$S_\omega = \frac{\pi^3\hbar c^3}{16F^2L^2\lambda P_\mathrm{in}} = \frac{\kappa}{16\bar{N}} \tag{10}$$

Where $\kappa = \pi c/LF$ is the cavity damping and $\bar{N}$ is the mean number of photons circulating in the cavity. This formula can be understood qualitatively by noting that during an observation time t a number $N \propto \bar{N}\kappa t$ of photons passes through the cavity which gives a shot noise limit $\delta\theta \propto 1/\sqrt{N} \propto \delta\omega/\kappa$ for the resolvable phase shift $\delta\theta$ (or the corresponding frequency shift $\delta\omega$). The spectral density in equation (10) then follows via $S_\omega \propto \delta\omega^2 t$.

The lifetime of a membrane phonon-number state *n* is limited by three effects. The first is the thermal lifetime given by[2]:

$$\tau_{\mathrm{T}} = \frac{Q}{\omega_m(n(\bar{n}+1)+\bar{n}(n+1))} \quad , \tag{11}$$

where the bath's mean phonon number $\bar{n} = k_B T/\hbar\omega_m$ (we assume $k_B T/\hbar\omega_m \gg 1$). If we also assume the membrane has been laser cooled to its ground state (n=0) then (11) becomes:

- $$\tau_{\mathrm{T}} = \frac{Q\hbar}{k_B T} \tag{12}$$

The second effect we consider is due to the terms discarded from $\hat{H}$ as a result of the RWA. These terms are:

$$\hat{N}\hbar\omega''_{\gamma,0}x_m^2(\hat{b}^\dagger\hat{b}^\dagger + \hat{b}\hat{b})/2 = \hat{N}\hbar\Delta\omega_\gamma(\hat{b}^\dagger\hat{b}^\dagger + \hat{b}\hat{b})/2\,. \tag{13}$$

Again, we assume that the membrane has been laser-cooled to its ground state. From Fermi's golden rule, the non-RWA terms will generate transitions from n = 0 to n = 2 at a rate:

$$R_{0\to 2} = \tfrac{1}{2}(\Delta\omega_\gamma)^2 S_{NN}(-2\omega_m)\,, \tag{14}$$

where [3]

$$S_{NN}(\omega) = \int dt \exp(i\omega t)\left\langle \hat{N}(t)\hat{N}(0)\right\rangle = \bar{N}\frac{\kappa}{(\omega+\Delta)^2 + (\kappa/2)^2} \tag{15}$$

is the photon shot noise (power) spectral density in the cavity and represents the power available via Raman processes to decrease the membrane's energy (for positive ω) or increase it (negative ω). Here Δ is the laser detuning relative to the cavity. Since we are considering displacement detection using a Pound-Drever-Hall detector, we take $\Delta = 0$ (i.e., the probe laser locked to the cavity). Therefore

$$S_{NN}(-2\omega_m) = \bar{N}\frac{\kappa}{(2\omega_m)^2 + (\kappa/2)^2}\,. \tag{16}$$

This gives

$$R_{0\to 2} = \frac{\left(\Delta\omega_\gamma\right)^2 \bar{N}}{8}\frac{\kappa}{\omega_m^2 + \kappa^2/16} \tag{17}$$

Plugging in our expression for $\Delta\omega_\gamma$ from above and using $\bar{N}\kappa = P_{in}\lambda/\pi\hbar c$ we have

- $$\tau_{RWA} = R_{0\to 2}^{-1} = \frac{\lambda^3 L^2 (1-r_c) m\omega_m (\omega_m^2 + \kappa^2/16)}{8\pi^3 x_m^2 c P_{in}} \tag{18}$$

Lastly, there is the excitation rate due to the membrane not being exactly at the extremum of the detuning curve shown in Fig. 1(e). This adds the following term to the Hamiltonian

$$\hat{N}\hbar\omega'_{\gamma,0}x_m(\hat{b}^\dagger + \hat{b}) . \tag{19}$$

Again, using Fermi's golden rule, this will generate transitions out of the membrane's ground state at a rate

$$R_{0\to1} = (\omega'_{\gamma,0}x_m)^2 S_{NN}(-\omega_m) \tag{20}$$

Using (15), we get:

- $$\tau_{lin} = R_{0\to1}^{-1} = \frac{m\omega_{\mathrm{m}}L^2\lambda^3(1-r_{\mathrm{c}})(4\omega_m^2+\kappa^2)}{256\pi^3P_{\mathrm{in}}cx_0^2} \tag{21}$$

The total lifetime of the membrane's ground state is then

- $$\tau^{(0)} = 1/(\tau_T^{-1} + \tau_{RWA}^{-1} + \tau_{lin}^{-1}) \tag{22}$$

The signal-to-noise ratio for observing a single quantum jump out of the membrane's ground state is then**:**

- $$SNR^{(0)} = (\Delta\omega_\gamma)^2\tau^{(0)}/S_\omega \tag{23}$$

Finally, we note that the different contributions to the lifetime obey the following relations (in the good cavity regime, where $\omega_{\mathrm{m}} \gg \kappa$, which is the most relevant regime).

The ratio of the total lifetime to the lifetime generated by a finite displacement is given by

$$\frac{\tau^{(0)}}{\tau_{\text{lin}}} = \frac{SNR^{(0)}}{16}\left(\frac{x_0}{x_{\text{m}}}\right)^2\left(\frac{\kappa}{\omega_{\text{m}}}\right)^2 \tag{24}$$

For the parameters used in the numerical estimates in our main paper (e.g., Table 1), the total lifetime is dominated by thermal transition and the ratio in (24) is small. Furthermore, the lifetime correction related to non-RWA effects is even smaller, since

$$\frac{\tau_{\text{lin}}}{\tau_{\text{RWA}}} = \frac{1}{8}\left(\frac{x_{\text{m}}}{x_{\text{o}}}\right)^2 \tag{25}$$

is much smaller than unity for reasonable estimates of the positioning accuracy x_0.

The parameters in Table 1 should readily allow for laser cooling to the membrane's ground state[3], as is assumed in these calculations. In addition, these parameters satisfy the condition $\tau^{(0)} > 1/\omega_{\text{m}}$, necessary for a QND measurement. Lastly, we note that when r_{c} approaches unity and x_0 approaches 0, two cavity modes approach degeneracy (as can be seen in Fig. 1(e) of our main paper). For our analysis to be valid, the gap Δ_{gap} between these two modes should be greater than ω_{m}. Since $\Delta_{\text{gap}} \approx (c/L)\sqrt{8(1-r_{\text{c}})}$, this conditioned is satisfied as long as 1-r_{c} > 10^{-8}.

Even if individual quantum jumps cannot easily be resolved, the approach outlined here can still be used to observe energy quantization in the membrane. Repeated measurements of the type described here (even with SNR < 1) could be converted to histograms and averaged together to reveal discretization of the membrane's energy.

While less dramatic than observations of individual quantum jumps, such an observation would still represent a major breakthrough.

1. Black, E. D. An introduction to Pound-Drever-Hall laser frequency stabilization. *Am. J. Phys.* **69**, 79 - 87 (2001).

2. D. H. Santamore, Doherty, A. C. & Cross, M. C. Quantum nondemolition measurements of Fock states of mesoscopic mechanical oscillators. *Phys. Rev. B* **70**, 144301 (2004).

3. Marquardt, F., Chen, J. P., Clerk, A. A. & Girvin, S. M. Quantum theory of cavity-assisted sideband cooling of mechanical motion. *Phys. Rev. Lett.* **99**, 093902 (2007).